BOUNDARY BEHAVIOR OF BMO(B_n)

BY

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ABSTRACT. If f is a holomorphic function of bounded mean oscillation in the unit ball of $\mathbf{C}^n$, then it has radial limits at almost every point of the boundary of the ball. The question remains as to how nicely one can expect this function to behave on subsets of the boundary of zero measure. For example, there is a holomorphic BMO function in the ball that has a finite radial limit at no point of the n-torus. We show here that this is not an isolated phenomenon; there exists at least one other n-dimensional submanifold of the boundary of the ball with this same behavior.

Let $B_n = \{ z \in \mathbf{C}^n : |z| < 1 \}$ be the unit ball of $\mathbf{C}^n$, and let σ_n denote the normalized Lebesgue measure on ∂B_n. We are interested in the following question: Given a submanifold M of ∂B_n and a function f holomorphic in B_n, what conditions on f will guarantee that f has radial limits almost everywhere on M? It is a classical result, see [6], for example, that if f is in the Hardy space $H^p(B_n)$, for $0 < p \leqslant \infty$, then f has radial limits almost everywhere on ∂B_n. If $f \in H^\infty(B_n)$, then more can be said. For example, Nagel and Rudin proved in [4] that if $\varphi: [a, b] \to B_n$ is a nowhere complex tangential curve and φ' satisfies a Lipschitz condition, then f has radial limits almost everywhere on φ with respect to arclength measure. It was recently shown by Ahern [1] that there exists a holomorphic function of bounded mean oscillation with radial limits at no point of the n-torus $T_n = \{ z \in \partial B_n : |z_j| = 1/\sqrt{n} \}$. Motivated by this, we show the existence of another n-dimensional submanifold of ∂B_n with this same behavior.

We define the holomorphic functions of bounded mean oscillation as follows. If $f \in H^2(B_n)$ we say that $f \in \mathrm{BMO}(B_n)$ if there exists a constant C such that for all $F \in H^2(B_n)$

$$\left| \int_{\partial B_n} F \bar{f} \, d\sigma_n \right| \leqslant C \| F \|_1.$$

It follows that $H^\infty(B_n) \subset \mathrm{BMO}(B_n) \subset H^p(B_n)$ for $0 < p < \infty$. (For more information on $\mathrm{BMO}(B_n)$ for $n > 1$, see [2].)

Let π be the mapping from $\mathbf{C}^n$ into $\mathbf{C}$ given by $\pi(z_1, z_2, \ldots, z_n) = \sum_{j=1}^{n} z_j^2$. One can easily verify that $\pi(B_n) = B_1$ and $\pi(\bar{B}_n) = \bar{B}_1$. We set $W_n = \pi^{-1}(\partial B_1)$. W_n is then a submanifold of ∂B_n with real dimension n, and it is this submanifold which we will consider. Note that for $n = 2$, W_n is biholomorphically equivalent to T_n via the mapping $(z_1, z_2) \to ((z_1 - iz_2)/\sqrt{2}, (z_1 + iz_2)/\sqrt{2})$.

To construct the desired function, we show that if g is holomorphic in B_1 with the property that $\sup_{|\lambda| < 1} |f'(\lambda)|(1 - |\lambda|) < \infty$, then $g \circ \pi \in \text{BMO}(B_n)$. (A function g with this property is called Bloch function.) It is known that there exists such a function G with radial limits at no point of ∂B_1 [5]. It follows that $G \circ \pi$ will have a radial limit at no point of $\pi^{-1}(\partial B_1) = W_n$. It thus remains to prove

THEOREM 1. *If g is a Bloch function, then $g \circ \pi \in \text{BMO}(B_n)$.*

We begin with a calculation which will be the main tool in the proof of Theorem 1.

THEOREM 2. *For each integer $n \geqslant 2$, there exists a constant K_n such that if f is a continuous complex-valued function on B_1, then*

(1) $$\int_{\partial B_n} f \circ \pi \, d\sigma_n = K_n \int_0^{2\pi} \int_0^1 f(re^{i\theta}) r(1 - r^2)^{(n-3)/2} \, dr \, d\theta.$$

We evaluate the left-hand side of (1) by applying the following reformulation of the co-area formula of Federer [3].

PROPOSITION. *If M and N are real Riemannian submanifolds, with $\dim_{\mathbf{R}} M = m \geqslant n = \dim_{\mathbf{R}} N$, and f is a smooth map, then*

(2) $$\int_M g(x) \, d\mu(x) = \int_N \left[\int_{f^{-1}\{y\}} \frac{g(x)}{J_n f(x)} \, dH^{m-n}(x) \right] dH^n(y),$$

where μ is Lebesgue measure on M, H^k is k-dimensional Hausdorff measure, and g is a Lebesgue integrable function on M satisfying the conditions
   (i)

$$\int_N \left[\int_{f^{-1}\{y\}} \left| \frac{g(x)}{J_n f(x)} \right| dH^{m-n}(x) \right] dH^n(y) < \infty$$

and
   (ii) *there exists $E_j \subset M$ such that $E_1 \subset E_2 \subset \cdots$, $\mu(\bigcup_{j=1}^\infty E_j) = 1$, and $(g/J_n f)\chi_{E_j} \in L^1(d\mu)$ for each j.*

Applying (2) to $M = S^{2n-1}$ and $N = B_1$ we obtain

$$\int_{S^{2n-1}} f \circ \pi \, d\sigma_n = C_n \int_{B_1} f(\lambda) \int_{\Gamma_\lambda} \frac{1}{J_2 \pi(x)} \, dH^{2n-3}(x) \, d\lambda,$$

where the constant C_n depends only on n, $\Gamma_\lambda = \pi^{-1}\{\lambda\} \cap S^{2n-1}$, and f is continuous on B_1. In polar coordinates this becomes

(3) $$\int_{S^{2n-1}} f \circ \pi \, d\sigma_n = C_n \int_0^{2\pi} \int_0^1 rf(re^{i\theta}) \left[\int_{\Gamma_{re^{i\theta}}} \frac{1}{J_2 \pi(x)} \, dH^{2n-3}(x) \right] dr \, d\theta.$$

Thus, our proof of Theorem 2 will be complete if we show that

(4) $$\int_{\Gamma_{re^{i\theta}}} \frac{1}{J_2 \pi(x)} \, dH^{2n-3}(x) = C_n(1 - r^2)^{(n-3)/2}.$$

This will be accomplished in the following lemmas.

LEMMA 1. $J_2\pi(\xi) = 4\sqrt{1 - |\pi(\xi)|^2}$.

PROOF. Since $J_2\pi = \frac{1}{2}\|d\pi \wedge d\bar{\pi}\|$, we must show that $\|d\pi(\xi) \wedge d\bar{\pi}(\xi)\| = 8\sqrt{1 - |\pi(\xi)|^2}$.

Let $\{t_j\}_{j=1}^{2n-1}$ be an orthonormal basis of $TS^{2n-1}(\xi)$, the real tangent space of S^{2n-1} at the point ξ. Let $\{w_j\}_{j=1}^{2n-1}$ be the corresponding dual basis of the cotangent space. Then we have $d\pi = \sum_{j=1}^{2n-1}(t_j\pi)w_j$. Hence, to prove the lemma, we construct such a basis and compute $d\pi$.

We choose our tangent vectors to be of the form

$$t_{2l}(\xi) = \mathrm{Re}\left(\sum_{k=1}^{n} \alpha_k^l(\xi)\frac{\partial}{\partial\xi_k}\right), \qquad 1 \leq l \leq n-1,$$

$$t_{2l+1}(\xi) = \mathrm{Im}\left(\sum_{k=1}^{n} \alpha_k^l(\xi)\frac{\partial}{\partial\xi_k}\right), \qquad 0 \leq l \leq n-1$$

for $\alpha^l \in \mathbf{C}^n$. One can easily check that the t_j's will be orthonormal if and only if

(5) $$|\alpha^l(\xi)| = 2 \quad\text{and}\quad \langle\alpha^l(\xi), \alpha^m(\xi)\rangle = 0.$$

Furthermore, in order for each $t_j(\xi)$ to be an element of $TS^{2n-1}(\xi)$, we must have $t_j(\sum_{k=1}^{n}|\xi_k|^2) = 0$. This implies that

(6) $$\langle\alpha^l(\xi), \xi\rangle = 0 \quad\text{for } 1 \leq l \leq n-1 \quad\text{and}\quad \mathrm{Im}\langle\alpha^0(\xi), \xi\rangle = 0.$$

Let $\xi \in S^{2n-1}$ so that $|\pi(\xi)| < 1$. Let $\alpha^0(\xi) = 2\xi$ and $\alpha^1(\xi) = 2q/|q|$, where $q = \bar{\xi} - \langle\bar{\xi}, \xi\rangle\xi$ is the component of $\bar{\xi}$ orthogonal to ξ. By (5) and (6) we see that $\{t_1(\xi), t_2(\xi), t_3(\xi)\}$ is an orthonormal set of tangent vectors for each $\xi \in S^{2n-1}$, with $|\pi(\xi)| < 1$.

We now choose $\alpha^3(\xi), \ldots, \alpha^{n-1}(\xi) \in \mathbf{C}^n$ so that (5) and (6) are satisfied. This will imply that $\alpha^l(\xi)$ is orthogonal to both $\alpha^1(\xi)$ and ξ for $2 \leq l \leq n-1$, and hence we have

(7) $$\langle\alpha^l(\xi), \bar{\xi}\rangle = 0, \qquad 2 \leq l \leq n-1.$$

To compute $d\pi$ we have

$$d\pi = \sum_{j=1}^{2n-1}(t_j\pi)w_j$$

$$= (t_1\pi)w_1 + \sum_{l=1}^{n-1}\left((t_{2l}\pi)w_{2l} + (t_{2l+1}\pi)w_{2l+1}\right)$$

$$= -i\langle\alpha^0, \bar{\xi}\rangle w_1 + \sum_{l=1}^{n-1}\left(\langle\alpha^l, \bar{\xi}\rangle w_{2l} - i\langle\alpha^l, \bar{\xi}\rangle w_{2l+1}\right),$$

which by (7) gives

$$d\pi = -i\langle\alpha^0, \bar{\xi}\rangle w_1 + \langle\alpha^1, \bar{\xi}\rangle w_2 - i\langle\alpha^1, \bar{\xi}\rangle w_3.$$

Hence,

$$d\pi \wedge d\bar{\pi} = -2i\left(\mathrm{Re}\left[\langle\alpha^0, \bar{\xi}\rangle\overline{\langle\alpha^1, \bar{\xi}\rangle}\right]w_1 \wedge w_2\right.$$

$$\left. - \mathrm{Im}\left[\langle\alpha^0, \bar{\xi}\rangle\overline{\langle\alpha^1, \bar{\xi}\rangle}\right]w_1 \wedge w_3 - \left|\langle\alpha^1, \bar{\xi}\rangle\right|^2 w_2 \wedge w_3\right)$$

and

$$\|d\pi \wedge d\bar{\pi}\|^2 = 4\left|\langle \alpha^1, \bar{\xi}\rangle\right|^2 \left(\left|\langle \alpha^0, \bar{\xi}\rangle\right|^2 + \left|\langle \alpha^1, \bar{\xi}\rangle\right|^2\right).$$

This gives

$$\|d\pi \wedge d\bar{\pi}\| = 2\left(2\sqrt{1 - |\pi(\xi)|^2}\right)\left(4|\pi(\xi)|^2 + 4\left(1 - |\pi(\xi)|^2\right)\right)^{1/2}$$

$$= 8\sqrt{1 - |\pi(\xi)|^2}.$$

Thus

$$J_2\pi(\xi) = 4\sqrt{1 - |\pi(\xi)|^2} \quad \text{for all } \xi \in S^{2n-1} \setminus W_n$$

and hence for all $\xi \in S^{2n-1}$. $\square$

LEMMA 2.

$$\int_{\Gamma_\lambda} dH^{2n-3} = C_n\left(1 - |\lambda|^2\right)^{(n-2)/2} \quad \text{for } 0 \leq |\lambda| < 1.$$

PROOF. Since $\Gamma_r = e^{i\theta/2}(\Gamma_{re^{i\theta}})$, and the volume of Γ_r, represented by the left-hand integral, is rotation invariant, it is sufficient to prove the lemma for $\lambda = r$.

Suppose $z = X + iY \in \Gamma_r$ where $X, Y \in \mathbf{R}^n$. It follows that $z \in \Gamma_r$ if and only if

$$(8) \qquad |X|^2 = \frac{1+r}{2}, \quad |Y|^2 = \frac{1-r}{2} \quad \text{and} \quad \langle X, Y\rangle = 0.$$

To evaluate $\int_{\Gamma_r} dH^{2n-3}$ we again use the co-area formula. We consider the map f: $\Gamma_r \to \sqrt{(1+r)/2}\, S^{n-1}$ defined by $f(X + iY) = X$. By the above comments we know that f maps Γ_r onto $\sqrt{(1+r)/2}\, S^{n-1}$. Thus, applying (2) for $g \equiv 1$, we have

$$(9) \qquad \int_{\Gamma_r} dH^{2n-3} = \int_{\sqrt{(1+r)/2}\, S^{n-1}} \left[\int_{f^{-1}\{p\}} \frac{1}{J_{n-1}f(x)}\, dH^{n-2}(x)\right] dH^{n-1}(p).$$

To compute $J_{n-1}f$, we construct an orthonormal basis of $T\Gamma_r(z)$ at $z = X + iY$.

For $1 \leq l \leq n - 2$ let $\alpha^l \in \mathbf{R}^n$ such that the set $\{\sqrt{2/(1-r)}\, Y, \alpha^1, \ldots, \alpha^{n-2}\}$ is an orthonormal basis of $TS^{n-1}(X)$. Note that this is possible since $\langle X, Y\rangle = 0$. Let

$$t_1 = \text{Im}\left[\frac{2}{\sqrt{1-r^2}} \sum_{j=1}^n (z_j - r\bar{z}_j)\frac{\partial}{\partial z_j}\right],$$

$$t_{2l} = \text{Re}\left[\sum_{j=1}^n 2\alpha^l_j\frac{\partial}{\partial z_j}\right], \quad t_{2l+1} = \text{Im}\left[\sum_{j=1}^n 2\alpha^l_j\frac{\partial}{\partial z_j}\right].$$

Because of our choice of the α^l's, it follows that the t_k's are orthonormal. To see that they are tangent vectors, we must show that

$$t_k\left(\sum_{j=1}^n z_j^2\right) = t_k\left(\sum_{j=1}^n z_j\bar{z}_j\right) = 0,$$

Since these are the defining functions of Γ_r. We have

$$t_1\left(\sum_{j=1}^n z_j^2\right) = \frac{-i}{\sqrt{1-r^2}} \sum_{j=1}^n (z_j - r\bar{z}_j)2z_j$$

$$= \frac{-i}{\sqrt{1-r^2}}(2\pi(z) - 2r|z|^2) = 0,$$

$$t_1\left(\sum_{j=1}^n z_j\bar{z}_j\right) = \frac{-i}{\sqrt{1-r^2}} \sum_{j=1}^n (z_j - r\bar{z}_j)\bar{z}_j - (\bar{z}_j - rz_j)z_j$$

$$= \frac{-ir}{\sqrt{1-r^2}}(\overline{-\pi(z)} + \pi(z)) = 0,$$

$$t_{2l}\left(\sum_{j=1}^n z_j^2\right) = \frac{1}{2}\sum_{j=1}^n (2\alpha_j')(2z_j) = 2\langle \alpha', \bar{z}\rangle = 0,$$

$$t_{2l}\left(\sum_{j=1}^n z_j\bar{z}_j\right) = \sum_{j=1}^n \alpha_j'\bar{z}_j + \alpha_j'z_j = 2\langle \alpha', X\rangle = 0$$

and similarly for t_{2l+1}. Thus the t_k's do form an orthonormal basis of $T\Gamma_r(z)$.

We now evaluate $t_k f_j$ where $f = (f_1, f_2, \ldots, f_n)$. We have

$$t_1 f_j = \sqrt{\frac{1+r}{1-r}}\, y_j; \quad t_{2l}f_j = \alpha_j'; \quad t_{2l+1}f_j = 0.$$

Thus, $\Lambda^{n-1}\, df$ is zero on each basis vector of $\Lambda_{n-1}(T\Gamma_r)$ with the exception of $t_1 \wedge t_2 \wedge t_4 \wedge \cdots \wedge t_{2n-4}$. Let S_j denote the permutation group on $\{1, 2, \ldots, j-1, j+1, \ldots, n\}$. If $\tau \in S_j$, let $\tau_1, \ldots, \tau_{n-1}$ denote the values $\tau(1), \tau(2), \ldots, \tau(j-1), \tau(j+1), \ldots, \tau(n)$, respectively. We have

$$\Lambda^{n-1}\, df(t_1 \wedge t_2 \wedge t_4 \wedge \cdots \wedge t_{2n-4})$$

$$= \sum_{j=1}^n \sum_{\tau \in S_j} (-1)^{|\tau|}(t_1 f_{\tau_1}) \cdots (t_{2n-4}f_{\tau_{n-1}})$$

$$= \sum_{j=1}^n \sum_{\tau \in S_j} (-1)^{|\tau|}\left(\sqrt{\frac{1+r}{1-r}}\right)y_{\tau_1}\alpha_{\tau_2}^1 \cdots \alpha_{\tau_{n-1}}^{n-2}$$

which is equal to $\sqrt{(1+r)/(1-r)}\,(Y \wedge \alpha^1 \wedge \cdots \wedge \alpha^{n-2})$. Hence,

$$J_{n-1}f = \|\Lambda^{n-1}\, df\| = \frac{1+r}{1-r}\|Y \wedge \alpha^1 \wedge \cdots \wedge \alpha^{n-2}\|$$

$$= \frac{\sqrt{1+r}}{2},$$

since $\{\sqrt{2/(1-r)}\, Y, \alpha^1, \ldots, \alpha^{n-2}\}$ is an orthonormal family. Thus (9) becomes

$$\int_{\Gamma_r} dH^{2n-3} = \int_{\sqrt{(1+r)/2}\, S^{n-1}}\left[\int_{f^{-1}\{p\}} \sqrt{\frac{2}{1+r}}\, dH^{n-2}(x)\right]dH^{n-1}(p).$$

Since $f^{-1}\{p\} = \{(p, Y): |Y| = \sqrt{(1-r)/2}, \langle p, Y \rangle = 0\}$ is an $(n-2)$-sphere we have

$$\int_{f^{-1}\{p\}} dH^{n-2} = C_n \left(\frac{1-r}{2} \right)^{(n-2)/2}.$$

Hence

$$\int_{\Gamma_r} dH^{2n-3} = C_n \left(\frac{1+r}{2} \right)^{(n-1)/2} (1+r)^{-1/2} \left(\frac{1-r}{2} \right)^{(n-2)/2}$$

$$= C_n (1-r^2)^{(n-2)/2}.$$

This proves Lemma 2, which together with Lemma 1 proves Theorem 2.

We now proceed to the proof of Theorem 1. It is sufficient to show that if g is a Bloch function on the unit disk, then there exists a constant $A = A(g)$ such that for all holomorphic polynomials we have

(10)
$$\left| \int_{\partial B_n} F(\overline{g \circ \pi}) \, d\sigma_n \right| \leqslant A \|F\|_1.$$

The method used to prove (10) follows closely that used by Ahern in [1].

Let $Tg = g \circ \pi$ and let $d\mu_n = K_n r(1-r^2)^{(n-2)/2} \, dr \, d\theta$, where K_n is the constant guaranteed by Theorem 3. It follows from this theorem that if g is continuous on B_1, then

$$\int_{\partial B_n} |Tg|^p \, d\sigma_n = \int_{B_1} |g|^p \, d\mu_n.$$

Thus T extends uniquely to be an isometry of $L^p(d\mu_n)$ into $L^p(d\sigma_n)$ for all finite p; in particular, for $p = 2$. This implies that if F is a holomorphic polynomial, then

$$\int_{\partial B_n} F(\overline{Tg}) \, d\sigma_n = \int_{B_1} (T^*F) \bar{g} \, d\mu_n$$

where T^* is the Hilbert space adjoint of the isometry from $L^2(d\mu_n)$ into $L^2(d\sigma_n)$. So, to prove (10), it is sufficient to prove

(11)
$$\left| \int_{B_1} (T^*F) \bar{g} \, d\mu_n \right| \leqslant A \|F\|_1$$

for all holomorphic polynomials F. This will be done in two steps. The first is to show that if F is a holomorphic polynomial in $\mathbf{C}^n$, then T^*F is a holomorphic polynomial in $\mathbf{C}$ and $\int_{B_1} |T^*F| \, d\mu_n \leqslant \int_{\partial B_n} |F| \, d\sigma_n$. The second step is to show that if g is a Bloch function then there exists a constant $A = A(g)$ such that for any holomorphic polynomial h of one variable we have

$$\left| \int_{B_1} h\bar{g} \, d\mu_n \right| \leqslant A \int_{B_1} |h| \, d\mu_n.$$

From these two steps the theorem clearly follows.

LEMMA 3. *If $F(z)$ is a holomorphic polynomial, then so is $T*F(\lambda)$. Moreover,* $\int_{B_1} |T*F| \, d\mu_n \leqslant \int_{\partial B_n} |F| \, d\sigma_n$.

PROOF. To prove the first part of the lemma, it is sufficient to consider only the holomorphic monomials z^{α}. We begin by making three observations:

(i) If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is any multi-index and $\tau(\alpha)$ is any permutation of the α_j's, then $T*(z^{\alpha}) = T*(z^{\tau(\alpha)})$. This follows from the fact that τ induces a unitary transformation on ∂B_n which leaves both π and σ_n invariant and so we have

$$\int_{\partial B_n} z^{\alpha} \left(\overline{g \circ \pi} \right) d\sigma_n = \int_{\partial B_n} z^{\tau(\alpha)} \left(\overline{g \circ \pi} \right) d\sigma_n,$$

from which the assertion follows.

(ii) $T*(z^{\alpha}) = 0$ if α_j is odd for any j. Because both σ_n and π are left invariant when z_j is replaced by $-z_j$ we have

$$\int_{\partial B_n} z^{\alpha} \left(\overline{g \circ \pi} \right) d\sigma_n = \int_{\partial B_n} (-1)^{\alpha_j} z^{\alpha} \left(\overline{g \circ \pi} \right) d\sigma_n$$

and thus $T*(z^{\alpha}) = 0$.

(iii)

$$T*\left(z_1^{2K} z_2^{2L} (z')^{2\beta} \right) = \frac{2K-1}{2L+1} T*\left(z_1^{2K-2} z_1^{2L+2} (z')^{2\beta} \right)$$

for all nonnegative integers K and L; $K > 0$, and all multi-indices β of the form $\beta = (\beta_1, \ldots, \beta_{n-2})$. (Note that $z' = (z_3, z_4, \ldots, z_n)$.)

We consider the unitary transformation

$$(z_1, z_2, \ldots, z_n) \rightarrow (z_1 \cos\theta + z_2 \sin\theta, \; z_1 \sin\theta - z_2 \cos\theta, \; z_3, \ldots, z_n)$$

for $\theta \in \mathbf{R}$. It is easily verified that π is invariant under this transformation, as is σ_n. Hence, we have

$$(12) \quad \int_{\partial B_n} z_1^s z_2^t (z')^{2\beta} \left(\overline{g \circ \pi} \right) d\sigma_n$$

$$= \int_{\partial B_n} (z_1 \cos\theta + z_2 \sin\theta)^s (z_1 \sin\theta - z_2 \cos\theta)^t (z')^{2\beta} \left(\overline{g \circ \pi} \right) d\sigma_n$$

for all nonnegative integers s and t and for all real θ. Differentiating both sides of (12) with respect to θ gives

$$0 = \int_{\partial B_n} (g \circ \pi)(z')^{2\beta} \Big[-s(z_1 \cos\theta + z_2 \sin\theta)^{s-1}(z_1 \sin\theta - z_2 \cos\theta)^{t+1}$$

$$+ t(z_1 \cos\theta + z_2 \sin\theta)^{s+1}(z_1 \sin\theta - z_2 \cos\theta)^{t-1} \Big] d\sigma_n.$$

By (12) this implies that

$$\int_{\partial B_n} s z_1^{s-1} z_2^{t+1} (z')^{2\beta} \left(\overline{g \circ \pi} \right) d\sigma_n = \int_{\partial B_n} t z_1^{s+1} z_2^{t-1} (z')^{2\beta} \left(\overline{g \circ \pi} \right) d\sigma_n.$$

Thus,

$$s T*\left(z_1^{s-1} z_2^{t+1} (z')^{2\beta} \right) = t T*\left(z_1^{s+1} z_2^{t-1} (z')^{2\beta} \right),$$

and choosing $s = 2K - 1$ and $t = 2L + 1$, we obtain our assertion.

For an arbitrary monomial z^α, it follows from (i) and successive applications of (iii) that

(13) $T^*(z^{2\alpha}) = C_{\alpha,\beta} T^*(z^{2\beta})$ for all β such that $|\beta| = |\alpha|$.

By Theorem 3 we know that

$$\int_{\partial B_n} [\pi(z)]^K (\overline{g \circ \pi}) \, d\sigma_n = \int_{B_1} \lambda^K \overline{g} \, d\mu_n,$$

and thus $T^*([\pi(z)]^K) = \lambda^K$. Expanding $[\pi(z)]^K$ gives

$$\lambda^K = T^*\left((z_1^2 + \cdots + z_n^2)^K\right) = \sum_{|\alpha|=K} \frac{K!}{\alpha!} T^*(z^{2\alpha});$$

hence by (13) we have

$$T^*(z^{2\alpha}) = C_\alpha \lambda^{|\alpha|}$$

for some constant C_α. This completes the proof of the first part of the lemma.

Since $T: L^p(d\mu_n) \to L^p(d\sigma_n)$, $1 \leqslant p < \infty$, is an isometry, it follows that $T^*: L^q(d\sigma_n) \to L^q(d\mu_n)$, $1 < q \leqslant \infty$, has norm at most 1. Thus, if F is a holomorphic polynomial in n variables it follows that

$$\int_{B_1} |T^*F|^q \, d\mu_n \leqslant \int_{\partial B_n} |F|^q \, d\sigma_n$$

for each $q > 1$. Letting $q \to 1$, we have, by the dominated convergence theorem,

$$\int_{B_1} |T^*F| \, d\mu_n = \lim_{q \to 1^+} \int_{B_1} |T^*F|^q \, d\mu_n$$

$$\leqslant \lim_{q \to 1^+} \int_{\partial B_n} |F|^q \, d\sigma_n = \int_{\partial B_n} |F| \, d\sigma_n. \quad \square$$

LEMMA 4. *Let g be a Bloch function in the unit disk. Then there exists a constant $A = A(g)$ such that, for any holomorphic polynomial h of one variable,*

(14) $$\left| \int_{B_1} h\overline{g} \, d\mu_n \right| \leqslant A \int_{B_1} |h| \, d\mu_n.$$

This is just Lemma 2.3 in [1] with $w_n(r) = K_n r (1 - r^2)^{(n-3)/2}$. This completes the final step of the proof of Theorem 2.

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